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1968 J. Phys. A: Gen. Phys. 1 690

(http://iopscience.iop.org/0022-3689/1/6/307)

J. PHYS. A (PROC. PHYS. SOC.), 1968, SER. 2, VOL. 1. PRINTED IN GREAT BRITAIN

Theory of periodic sampling of photon-counting distributions

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MS. received 30th May 1968

Abstract. The statistical accuracy with which moments of the intensity-fluctuation distributions of optical fields can be determined, by photon-counting experiments of finite duration, is calculated.

1. Introduction

The probability distributions $P(E)$ of fluctuations in the integrated intensities

$$E(t, T) = \int_{t-T/2}^{t+T/2} \mathscr{E}^{-}(\boldsymbol{r}t')\mathscr{E}^{+}(\boldsymbol{r}t')\,\mathrm{d}t' \tag{1}$$

of optical fields $\mathscr{E}(\boldsymbol{r}, t)$ have been the subject of a number of recent experimental and theoretical investigations. $\mathscr{E}$ is a scalar component of the electric-field vector, the superscripts plus and minus indicate the positive- and negative-frequency parts, respectively, $\boldsymbol{r}$ and t are space and time variables and T is an integration time. These distributions have been used both to characterize coherence properties of light sources and to measure correlations in scattering media. Theoretical reviews have been given by Glauber (1965) and by Mandel and Wolf (1965). Experiments usually measure fluctuations of $E(T)$ by taking periodic samples of the number n of photoelectrons emitted in a time T by a photoelectric detector (photomultiplier). This gives a photon-counting distribution $p(n, T)$ whose factorial moments defined by

$$N^{(r)} = \sum_{n=r}^{\infty} n(n-1)\ldots(n-r+1)p(n, T) \tag{2}$$

are α^r times the actual moments

$$\langle E^r \rangle = \int_0^{+\infty} E^r P(E)\,\mathrm{d}E \tag{3}$$

of the intensity-fluctuation distribution, where α is the quantum efficiency of the detector. This relation can be easily proved with the aid of Mandel's (1963) formula

$$p(n, T) = \int_0^{\infty} \frac{(\alpha E)^n}{n!}\,\mathrm{e}^{-\alpha E} P(E)\,\mathrm{d}E.$$

The first measurements of complete photon-counting distributions were, we think, by Johnson *et al.* (1965), although previous authors had made use of the variance of the distribution, sometimes called the excess noise. Complete distributions of a similar nature for α-particle detections were published by Rutherford and Geiger (1910), and similar work must also exist in the literature on the statistics of x-ray photons. Recent experimental work includes contributions by Arecchi *et al.* (1967), Chang *et al.* (1968) and Jakeman *et al.* (1968 a, b).

In a single experiment of M samples, an (unbiased) estimate

$$\begin{aligned}\mathscr{N}^{(r)} &= \sum_{n=r}^{\infty} n(n-1)\ldots(n-r+1)p(n) \\ &= \sum_{n=r}^{\infty} n(n-1)\ldots(n-r+1)\frac{1}{M}\sum_{p=1}^{M}\delta_{n_p n} \\ &= \frac{1}{M}\sum_{p=1}^{M} n_p(n_p-1)\ldots(n_p-r+1)\end{aligned} \tag{4}$$

of the factorial moment $N^{(r)}$ is obtained, where n_p is the number of counts in the pth sample, $\delta_{n_p n}$ is the Kronecker delta. In this paper the theoretical deviation of $\mathcal{N}^{(r)}$ from $N^{(r)}$ is calculated as a function of M. A general expression for the variance of the estimator $\mathcal{N}^{(r)}$ is obtained in terms of the correlation functions of the field and this is then evaluated for certain situations of experimental interest.

2. Variance of the estimator

The variance of $\mathcal{N}^{(r)}$ defined by equation (4) is given by

$$\operatorname{var}\mathcal{N}^{(r)} = \frac{1}{M^2} \sum_{p,q=1}^{M} [\langle n_p(n_p-1)\dots(n_p-r+1)n_q(n_q-1)\dots(n_q-r+1)\rangle - \{N^{(r)}\}^2]. \quad (5)$$

For random sampling at intervals greater than any coherence time of the field, the terms with p different from q vanish, while the rest are all equal. Making use of the identity

$$\{n(n-1)\dots(n-r+1)\}^2 \equiv \sum_{s=0}^{r} s! \binom{r}{s}^2 n(n-1)\dots(n-2r+s+1) \quad (6)$$

these remaining terms may be summed to give

$$\operatorname{var}\mathcal{N}_{\mathrm{R}}^{(r)} = \frac{1}{M}\left[\sum_{s=0}^{r} s! \binom{r}{s}^2 N^{(2r-s)} - \{N^{(r)}\}^2\right] \quad (7)$$

where the suffix R denotes random sampling. The general formula (5) may now be expressed in terms of var $\mathcal{N}_{\mathrm{R}}^{(r)}$ and the 'off-diagonal' elements occurring in the sum. The latter may be rewritten using the relation

$$\langle n_p(n_p-1)\dots(n_p-r+1)n_q(n_q-1)\dots(n_q-r+1)\rangle = \alpha^{2r}\langle E_p{}^r E_q{}^r\rangle, \qquad p \neq q \quad (8)$$

and (5) becomes

$$\operatorname{var}\mathcal{N}^{(r)} = \operatorname{var}\mathcal{N}_{\mathrm{R}}^{(r)} + \frac{\alpha^{2r}}{M^2}\sum_{p\neq q}(\langle E_p{}^r E_q{}^r\rangle - \langle E^r\rangle^2). \quad (9)$$

The autocorrelation function of E^r may be expressed in terms of the correlation functions of the field, defined by Glauber (1963 a) as follows:

$$\langle E_p{}^r E_q{}^r\rangle = \int_{t_p-T/2}^{t_p+T/2} dt_1 \dots dt_r \int_{t_q-T/2}^{t_q+T/2} dt_{r+1}\dots dt_{2r} G^{(2r)}(t_1\dots t_{2r}, t_{2r}\dots t_1) \quad (10)$$

where the G are defined in terms of the density matrix ρ of the field by

$$G^{(n)}(t_1\dots t_{2n}) = \operatorname{Tr}\{\rho\hat{\mathscr{E}}^-(t)\dots\hat{\mathscr{E}}^-(t_n)\hat{\mathscr{E}}^+(t_{n+1})\dots\hat{\mathscr{E}}^+(t_{2n})\};$$

the caret denotes the field operator.

3. Examples

For periodic sampling of a stationary process (Davenport and Root 1958), which is the situation encountered in typical photon-counting experiments, the general expression (9) reduces to the form

$$\operatorname{var}\mathcal{N}^{(r)} = \operatorname{var}\mathcal{N}_{\mathrm{R}}^{(r)} + \frac{2\{N^{(r)}\}^2}{M}\sum_{k=1}^{M-1}\left(1-\frac{k}{M}\right)\left(\frac{\langle E_p{}^r E_{p+k}{}^r\rangle}{\langle E^r\rangle^2} - 1\right). \quad (11)$$

This will first be evaluated for the case of a fully coherent field. The correlation functions then factorize (Glauber 1963 a):

$$G^{(n)}(t_1 \ldots t_n, t_n \ldots t_1) = \prod_{j=1}^{n} G^{(1)}(t_j, t_j) \tag{12}$$

and the multiple integrals on the right-hand side of equation (10) factorize in a similar way to give

$$\langle E_p{}^r E_{p+k}{}^r \rangle = \langle E^r \rangle^2 = \langle E^{2r} \rangle. \tag{13}$$

The second term of (11) then vanishes and, writing $\bar{n}$ for $N^{(1)}$ and using (7), this equation reduces to

$$\mathrm{var}\mathcal{N}^{(r)} = \mathrm{var}\mathcal{N}_{\mathrm{R}}{}^{(r)} = \frac{1}{M}\sum_{s=1}^{r} s!\binom{r}{s}^2 \bar{n}^{2r-s}. \tag{14}$$

As a second example we take the case of Gaussian–Lorentzian light (Jakeman and Pike 1968) whose correlation time τ_c satisfies the double inequality $M\tau \gg \tau_c \gg r\tau$ where $\tau > T$ is the period of the sampling. For Gaussian light, the correlation functions possess the following properties (Glauber 1963 b):

$$G^n(t_1 \ldots t_n, t_n \ldots t_1) = \sum_{\mathrm{perm}} \prod_{j=1}^{n} G^{(1)}(t_j, t_{\mathrm{perm}(n+j)}). \tag{15}$$

If the spectrum is Lorentzian $G^{(1)}(t_p, t_q)$ is given by (Glauber 1963 b)

$$G^{(1)}(t_p, t_q) = \frac{\langle E \rangle}{T}\exp\{\mathrm{i}\omega_0(t_p - t_q)\}\exp\left(-\frac{|t_p - t_q|}{\tau_c}\right) \tag{16}$$

and in the limit $\tau_c \gg T$ evaluation of the integrals in equation (10) leads to the relation

$$\langle E_p{}^r E_{p+k}{}^r \rangle = \langle E^r \rangle (r!)^2 \sum_{s=0}^{r} \binom{r}{s}^2 \exp\left(\frac{2k\tau s}{\tau_c}\right). \tag{17}$$

Substituting into (11) and summing over k gives

$$\mathrm{var}\mathcal{N}^{(r)} = \mathrm{var}\mathcal{N}_{\mathrm{R}}{}^{(r)} + \frac{2(r!)^2}{M}\bar{n}^{2r}\sum_{s=1}^{r}\binom{r}{s}^2 \times \left[\frac{M\{1-\exp(-2\tau s/\tau_c)\} + \exp(-2\tau s/\tau_c)\{\exp(-2M\tau s/\tau_c) - 1\}}{M\{\exp(-2\tau s/\tau_c) - 1\}^2}\right]. \tag{18}$$

In the same limit, the first term on the right-hand side of this equation may be written

$$\mathrm{var}\mathcal{N}_{\mathrm{R}}{}^{(r)} = \frac{1}{M}\sum_{s=0}^{r}\left\{s!\,(2r-s)!\binom{r}{s}^2 \bar{n}^{2r-s} - (r!)^2\bar{n}^{2r}\right\}. \tag{19}$$

If $\tau r/\tau_c \ll 1$ the second term on the right-hand side of (18) dominates the first term. If, in addition, the total experiment time $M\tau$ is much greater than τ_c, (18) is well approximated by the formula

$$\mathrm{var}\,\mathcal{N}^{(r)} = \frac{\tau_c}{\tau}\frac{\bar{n}^{2r}(r!)^2}{M}\sum_{s=1}^{r}\frac{1}{s}\binom{r}{s}^2. \tag{20}$$

The application of equations (14) and (20) is reported elsewhere (Jakeman *et al.* 1968 a, b). They have been found to agree well with experiment up to the sixth order.

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